Introductio

in Elementarem Functionum Ellipticarum Theoriam,

quam

Cons. Ampliss. Philos. Ord. Lundens.,

Præside

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Pro Laurea

p. p.

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LUNDÆ,

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Sect. II. Mutatio moduli dyadica.

§ Ex eis, quæ in sectione prima exposuimus, facile colligitur, functiones Ellipticas (primi generis) ad unicam fere novam functionem transcendentem, *amplitudinem* dico ($\mathcal{A}$), ejusque inversam (F), redigi. Vidimus enim (p. 39), ipsarum plerasque per amplitudines α & β harumque functiones exprimi. Jam vero invenimus alteram amplitudinem $\alpha = \mathcal{A}(A) - \mathcal{A}(A - F\beta) = \frac{1}{2}\pi - \mathcal{A}(A - F\beta)$, ideoque hujus complementum $\alpha' = \frac{1}{2}\pi - \alpha = \mathcal{A}(A - F\beta)$ seu $\alpha' = \mathcal{A}\,{}^{\text{c}}F\beta$, h. est: complementum trigonometricum α' hujus amplitudinis α esse co- amplitudinem functionis *nomicæ* (i. e. ejus quæ amplitudinis inversa est) illius amplitudinis β. Erit igitur $F(\frac{1}{2}\pi - \alpha) = A - F\beta = F(\frac{1}{2}\pi) - F\beta$, siquidem universim fuerit $\int \frac{d\psi}{\sqrt{a^2 - c^2\,\overline{S^0\psi}^2}} = F\psi$, cujus inversa ipsa est amplitudo $\mathcal{A}$, scilicet si $F\psi = x$ fuerit, erit inverse $\psi = \mathcal{A}\,x$. Facile vero reperitur $\mathcal{A}\,\psi = \mathfrak{z}\,(a, b, cS^0\psi)$ *). Sin vero $a = 1$ acceperimus, quod quidem ex eis, quæ p. 40 annotavimus, semper licet, erit functio F *nomica* seu ipsa illa primi generis celeberrimi Domini Le Gendre.

Si igitur tabulam construxeris (qualis Hujus jam est), quæ ad angulum vel amplitudinem quamcumque ψ suppeditaverit functionem $F\psi = x$, ideoque inverse amplitudinem ψ ($= \mathcal{A}\,x$) ad quemcumque illius integralis valorem x; hujus ope ad amplitudinis majoris quemvis valorem β invenies hanc functionem $F\beta$, hujusque complementum ellipticum $A - F\beta = {}^{\text{c}}F_\beta = F(\frac{1}{2}\pi) - F\beta$, (ubi $F(\frac{1}{2}\pi)$ idem ac F^1c Domini Le Gendre valet, si $a = 1$), deinde inverse habebis hujus amplitudinem $\mathcal{A}\,{}^{\text{c}}F_\beta$, quæ jam est ($= \alpha' = \frac{1}{2}\pi - \alpha$) complementum trigonometricum amplitudinis minoris (α) **). Quo facto, omnes functiones el-

*) Signum $\mathfrak{z}$ loco ipsius ς seu inversæ S adhiberi potest.

**) Quin imo ex formula $a\,T^0\alpha = b.T^0\beta$ obtinetur α ex β, sc. minor amplitudo

lipticæ, præter *subtangentes*, (quæ & ideo nonnullis *differentium* nomine veniunt), per unicam amplitudinem & functiones trigonometricas exprimentur. Nihilo tamen minus commodius censemus, si, sicuti jam p. 40 innuimus, utraque amplitudo una cum subtangente (seu *differente*) in tabula inveniretur; Ex. gr. sub hac forma:

Sector	Amplitudo major *)	(differens seu) subtangens	co-ampl. maj. s. complement. amp. min.	cosector
φ	$\beta = \mathfrak{A}\,\varphi \left(= {}^{0}\mathfrak{L}\,\frac{T\varphi}{b}\right.$ seu Arc. sin $\left.\left(\frac{S\varphi}{c}\right)\right)$	$\mathfrak{T}\,\varphi \left(= a\,.\,\frac{Sin\,\alpha}{Sin\,\beta}\right.$ seu $\left.= a\,.\,\frac{Cos\,\alpha'}{Sin\,\beta}\right)$	$\mathfrak{A}^{c}\varphi\,(=\alpha' = \mathfrak{A}\,(A-\varphi))$	${}^{c}\varphi (= A - \varphi)$
			ampl. maj. cosect.	

Tabula hujusmodi ad valorem $a = 1$ computata omnibus casibus sufficit. Tum vero $c = e$ evadit & omnes valores ab $e = o$ usque ad $e = 1$ adsciscere potest, sicuti φ omnes a $\varphi = o$ usque ad $\varphi = A = F\,(e, \frac{1}{2}\pi)$. Quoniam vero tabula ita justo major evaderet, videamus, numne minui possit.

Quem in finem vestigiis jam pervagatis pressius insistamus. Ex eis nempe quæ p. 53 & 54 proposuimus, radiorum principalium, ideoque

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ex majori, per solas functiones trigonometricas, sc. $\alpha = \mathfrak{L}^{0}\left(\frac{b}{a}.\,T^{0}\,\beta\right)$. Igitur & cyclice & hypercyclice ad eandem ipsius α valorem pervenire licet, sicuti in mari per diversas vias ad eundem locum appellimus. Inde vero colligitur hæc formula $a = b\,T^{0}\,\beta\,.\,T^{0}\,(\mathfrak{A}^{c}F\beta)$, & simul reliquas quantitatis $\mathfrak{A}^{c}F\beta$ functiones trigonometricas per $T^{0}\beta$ algebraice exprimas.

*) Æquo jure amplitudo minor α adhiberi posset, vel hujus complementum α'. Est scilicet $\beta = \mathfrak{A}^{c}F\alpha'$ seu β coamplitudo ipsius $F\alpha'$, quod si $= p$ ponitur, erit $\alpha' = \mathfrak{A}\,p$, $\beta = \mathfrak{A}^{c}\,p = \mathfrak{A}\,(A - p)$.

& moduli, mutationem colligere licet. Posito enim $R^4 + 2\varepsilon R^2 T^2 + T^4 = (a^2 + T^2).\ (b^2 + T^2)$, erit $a^2 b^2 = R^4$ & $a^2 + b^2 = 2\varepsilon R^2$, indeque a^2 vel $b^2 = R^2 (\pm\sqrt{\varepsilon^2 - 1} + \varepsilon)$. Deinde posito $R_{,}^4 + 2\varepsilon' R_{,}^2 T'^2 + T'^4 = (a_{,}^2 + T'^2)(b_{,}^2 + T'^2)$, erit similiter $a_{,}^2$ vel $b_{,}^2 = R_{,}^2 (\varepsilon_{,} \pm \sqrt{\varepsilon_{,}^2 - 1})$; ideoque (si $T' = r_{,} . \dfrac{R-T}{R+T}$) radii $a_{,}$ & $b_{,}$ in a & b mutabuntur. Erit vero simul tertius radius $= c = \sqrt{a^2 - b^2} = R\sqrt{2}.\sqrt[4]{\varepsilon^2 - 1}$, atque $c' = \sqrt{a_{,}^2 - b_{,}^2} = R\sqrt{2}.\sqrt[4]{\varepsilon'^2 - 1}$, & igitur modulus $e = \dfrac{c}{a} = \dfrac{\sqrt{2}.\sqrt[4]{\varepsilon^2 - 1}}{\sqrt{\varepsilon + \sqrt{\varepsilon^2 - 1}}}$, atque alter $e' = \dfrac{c_{,}}{a_{,}} = \dfrac{\sqrt{2}.\sqrt[4]{\varepsilon_{,}^2 - 1}}{\sqrt{\varepsilon_{,} + \sqrt{\varepsilon_{,}^2 - 1}}}$. Loco vero dicto vidimus $\varepsilon' = \dfrac{3 - \varepsilon}{1 + \varepsilon}$ esse, ideoque, elisis ε & ε', modulorum e & e' alter per alterum exprimitur. Calculo rite subducto invenitur $\pm\sqrt{e'}$ vel $\pm\sqrt{\dfrac{1}{e'}} = \dfrac{\sqrt{e} - 1}{\sqrt{e} + 1}$.

Jam vero si ε inter limites -1 & -3 accipitur, (ex. gr. $-\varepsilon = 1 + w$, & $w < 2$,) erit $\varepsilon' \left(= \dfrac{4 + w}{-w}\right)$ inter limites $-\infty$ & -3; at ratio $\dfrac{R}{r_{,}} = \sqrt[4]{\dfrac{1+\varepsilon}{2}} \left(= \sqrt[4]{\dfrac{-w}{2}}\right)$ imaginaria evadet. Nihilo tamen minus hæc transformatio in realem mutatur. Sit enim $r' = 1$, $\varepsilon = -\eta$, & $\varepsilon' = -\eta'$, ideoque formula proposita $\dfrac{dx}{\sqrt{1 - 2\eta' x^2 + x^4}}$ in $= \dfrac{-R^3\, dT}{\sqrt{R^4 - 2\eta R^2 T^2 + T^4}}$ mutata, ponendo $x (= T') = \dfrac{R-T}{R+T}$ et

$\eta' = \frac{\eta+3}{\eta-1}$, atque $R = \sqrt[4]{\frac{1-\eta}{2}}$. Sin vero $\eta > 1$ fuerit, erit etiam $\eta' > 1$, ideoque utrorumque denominatorum factores reales. Quia vero tum R imaginarium evadit, ponamus tantummodo $x = \frac{1-u}{1+u}$, eritque $\frac{dx}{\sqrt{1-2\eta' x^2 + x^4}} = \frac{-2\,du}{\sqrt{2.(1-\eta').(1-2\eta u^2 + u^4)}}$, (quia $T = Ru$ est, atque $\overline{\eta-1}.\overline{\eta'-1} = 4$). Resolvamus vero $(1-2\eta' x^2 + x^4)$ in $=(\varkappa^2 - x^2).(\varkappa^{-2} - x^2)$ atque $1-2\eta u^2 + u^4$ in $(\varkappa'^2 - u^2)(\varkappa'^{-2} - u^2)$, eritque $2\eta' = \varkappa^2 + \varkappa^{-2}$ atque $2\eta = \varkappa'^2 + \varkappa'^{-2}$, & $\pm\sqrt{2(\eta'-1)} = \varkappa - \varkappa^{-1}$ & $\pm\sqrt{2(\eta-1)} = \varkappa' - \varkappa'^{-1}$, & inde $(\varkappa^{-1} - \varkappa)(\varkappa'^{-1} - \varkappa') = \pm\sqrt{2.2(\eta'-1).(\eta-1)} = \pm 4$. Hanc vero æquationem resolvendo obtinetur $\varkappa' = \frac{\pm 2\varkappa \pm (1+\varkappa^2)}{1-\varkappa^2} = \frac{\pm\overline{1\pm\varkappa}^2}{1-\varkappa^2}$, ideoque $\pm\varkappa'$ vel $= \frac{1-\varkappa}{1+\varkappa}$ vel $= \frac{1+\varkappa}{1-\varkappa}$. Ponamus $\varkappa < \varkappa^{-1}$ (radii enim inæquales sunt) seu $\varkappa^2 < 1$, & $\varkappa < 1$, similiterque poni poterit $\varkappa' < \varkappa'^{-1}$ seu $\varkappa' < 1$, ideoque erit $\varkappa' = \frac{1-\varkappa}{1+\varkappa}$; quæ jam est modulorum ratio in transformatione ista imaginaria (seu generalior hæc $\pm\varkappa' = \left(\frac{1\pm\varkappa}{1\pm\varkappa}\right)$). Ut vero realem obtineamus, ponamus $\varkappa'^{-2} - u^2 = s^2$, eritque $-u\,du = s\,ds$, $u^2 = \varkappa'^{-2} - s^2$, $2(1-\eta') = -(\varkappa^2 + \varkappa^{-2} - 2) = -(\varkappa^{-1} - \varkappa)^2$, $\varkappa'^2 - u^2 = \varkappa'^2 - \varkappa'^{-2} + s^2 =$ $= -(\varkappa'^{-2} - \varkappa'^2 - s^2)$, ideoque $2(1-\eta').(1-2\eta u^2 + u^4) =$ $= \overline{\varkappa^{-1} - \varkappa}^2.(u^2 - \varkappa'^2).s^2 = \overline{\varkappa^{-1} - \varkappa}^2.s^2.\overline{\beta^2 - s^2}$, siquidem $\beta^2 =$ $= \varkappa'^{-2} - \varkappa'^2$ fuerit, & igitur

$$\frac{dx}{\sqrt{(\varkappa^2 - x^2)(\varkappa^{-2} - x^2)}} = \frac{2}{\varkappa^{-1} - \varkappa} \cdot \frac{ds}{\sqrt{(\beta^2 - s^2).(\varkappa'^{-2} - s^2)}}.$$ Sin vero ponimus $s = \frac{x'}{n}$ & radicem posterioris denominatoris

$$= \sqrt{k^2 (\widehat{\beta n}^2 - x'^2)(n^2 \varkappa'^{-2} - x'^2)} = \sqrt{k^2 . (k^2 - x'^2)(k^{-2} - x'^2)},$$ erit

$\widehat{\beta n}^2 = k^2$ & $n^2 = \varkappa'^2 . k^{-2}$, ideoque $n^2 = \frac{\varkappa'}{\beta}$ atque $\widehat{\beta n}^2 . \varkappa'^2 k^{-2} = k^2 . n^2$,

seu $k^4 = \beta^2 \varkappa'^2 = 1 - \varkappa'^4$, & $k^4 = \frac{\overline{1+\varkappa}^4 - \overline{1-\varkappa}^4}{\overline{1+\varkappa}^4} = \frac{8(\varkappa + \varkappa^3)}{\overline{1+\varkappa}^4}$, atque igitur $k = \frac{\sqrt[4]{8\varkappa(1+\varkappa^2)}}{1+\varkappa}$, quæ nova est modulorum ratio, eaque realis, sicuti argumentorum. Habemus enim $x = \frac{1-u}{1+u}$ atque $u =$

$$= \sqrt{\varkappa'^{-2} - \frac{\beta x'^2}{\varkappa'}} = \frac{\sqrt{1 - \varkappa'\beta x'^2}}{\varkappa'}$$ seu $u = \frac{\sqrt{1 - k^2 x'^2}}{\varkappa'}$, ideoque

$$x = \frac{\varkappa' - \sqrt{1 - k^2 x'^2}}{\varkappa' + \sqrt{1 - k^2 x'^2}}$$ atque $$\frac{(\varkappa^{-1} - \varkappa)\, dx}{\sqrt{(\varkappa^2 - x^2)(\varkappa^{-2} - x^2)}} =$$

$$= \frac{2\, dx'.n}{\sqrt{(k^2 - x'^2).(k^{-2} - x'^2)}} = \frac{2\varkappa'}{k} \frac{dx'}{\sqrt{(k^2 - x'^2)(k^{-2} - x'^2)}},$$ quæ formula semper realis est, quamdiu $x < \varkappa < 1$; nam positio $x = o$ facit $x' = \frac{\sqrt{1 - \varkappa'^2}}{k}$, & $x = \varkappa$ præbet $x' = k$ (vel o), quare $x' < k$, quamdiu $x < \varkappa$. Erit igitur integrando

$$\frac{1 - \varkappa^2}{\varkappa} \cdot S\,\overline{\frac{a}{b}},\ x = \frac{2\varkappa'}{k} \cdot \left(S\,\overline{\frac{a'}{b'}},\ x' - S\,\overline{\frac{a'}{b'},\ \frac{\sqrt{1 - \varkappa'^2}}{k}} \right),$$

seu $(\varkappa^{-1}-\varkappa).[S(a,b,x)-S(a,b,\varkappa)]=\frac{2\varkappa'}{k}.S(a',b',x')$, existentibus radiis $a=\varkappa^{-1}$, $c=\varkappa$, $b=\sqrt{a^2-c^2}=\frac{\sqrt{1-\varkappa^4}}{\varkappa}$, atque $a'=k^{-1}, c'=k$, $b'=\sqrt{a'^2-c'^2}=\frac{\sqrt{1-k^4}}{k}=\frac{\varkappa'^2}{k}$, ideoque modulis $\frac{c}{a}=\varkappa^2$ & $\frac{c'}{a'}=$ $=k^2=\frac{\sqrt{8\varkappa(1+\varkappa^2)}}{(1+\varkappa)^2}$, seu altero e $(=\varkappa^2)$, alteroque $e'=\frac{\sqrt{8\sqrt{e}.(1+e)}}{(1+\sqrt{e})^2}$.

Hic vero valor ejusmodi est, ut unica adhibita substitutione

$$x=\frac{\varkappa'-\sqrt{1-k^2z^2}}{\varkappa'+\sqrt{1-k^2z^2}}\text{ vel hujus inversa } z=\frac{\sqrt{1-\varkappa'^2}}{k}.\frac{\sqrt{1+2\mathfrak{K}x+x^2}}{1+x},$$

ubi $\mathfrak{K}=\frac{1+\varkappa'^2}{1-\varkappa'^2}$, functiones nostræ ellipticæ eo redigantur, ut modulus vel quam minimus evadat vel etiam unitati quam proxime accedat. Posito enim $e=0{,}0009$, erit $e'=\frac{\sqrt{8.0{,}03.1{,}0009}}{(1{,}03)^2}=0{,}4629$

$$e=0{,}0025,\ \ldots\ e'=\frac{\sqrt{8.0{,}05.1{,}0025}}{(1.05)^2}=0{,}5743$$

$$e=0{,}0049,\ \ldots\ e'=\frac{\sqrt{8.0{,}07.1{,}0049}}{(1{,}07)^2}=0{,}6552$$

& si $e=0{,}0074697$, erit $e'=\ldots\ldots\ldots\ldots=0{,}7071..=\sqrt{\tfrac{1}{2}}$; ideoque datis functionibus, quarum moduli $<0{,}0009$, vel $<0{,}005$ sunt, habebuntur & valores earum, quarum moduli $<0{,}4\ldots$ vel $<0{,}6\ldots$ sunt. Sit deinde $e=0{,}36$, eritque $e'=0{,}9973$ & si $e=0{,}49$, erit $1-e'=0{,}000485$, atque $1-e'=0{,}000076$ si $e=0{,}64$ & tandem si... $e=\sqrt{\tfrac{1}{2}}$, erit $1-e'=0{,}000028$; quare functiones, quibus moduli

sunt $> 0{,}36$ vel $> 0{,}49$ aut $> \sqrt{\frac{1}{2}}$, in alias mutantur, in quibus moduli differunt minus ab unitate, quam cum fractione 0,003 vel 0,00048 aut 0,000028. In unoquoque igitur casu, unica hujusmodi adhibita transformatione, functio hypercyclica (ς) moduli cujuscunque per aliam ejusmodi exprimetur, in qua modulus e vel $<$ 0,00747 vel hujus complementum $1-e <$ 0,000028.. erit. Quæ cum ita sint, vel minima tabulæ hypercyclicæ pars sufficiet, vel etiam hac ipsa carere fere licebit, cum pro tam parvis harum quantitatum valoribus formula differentialis facile in seriem convergentissimam convertatur, quæ et sine labore integretur *).

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*) Vulgari modo evolvendo & integrando obtinemus sc.

1:o) $$\int \frac{d\psi}{\sqrt{1-c^2 S^\circ \psi^2}} = \int d\psi \left(1 + \tfrac{1}{2} c^2 S^\circ \psi^2 + \tfrac{1}{2}.\tfrac{3}{4} c^4 S^\circ \psi^4 + \&\ldots\right)$$
$$= \psi + \frac{c^2}{2} \int S^\circ \psi^2 d\psi + \frac{3}{8}.c^4 \int S^\circ \psi^4 d\psi \& \ldots,$$
ubi $$\int S^\circ \psi^2 d\psi = \int d\psi . \frac{1 - C^\circ 2\psi}{2} = \frac{\psi}{2} - \frac{S^\circ 2\psi}{4}$$
& $$\int d\psi S^\circ \psi^4 = \frac{3}{8} \psi - \frac{1}{4} S^\circ 2\psi + \frac{1}{32} S^\circ 4\psi.$$

Et tres jam termini ad XIV locos decimales computandos sufficient, si $c^2 < 0{,}000056$ (seu $c < 0{,}0075$); & hoc casu duo priores ad IX usque locos accuratam dabunt functionem F (c, ψ).

2:o) $$\int \frac{d\psi}{\sqrt{1-c^2(1-C^\circ \psi^2)}} = \int \frac{d\psi}{\sqrt{c^2 C^\circ \psi^2 + b^2}} =$$
$$= c^{-1} \int \frac{d\psi}{C^\circ \psi} - \frac{1}{2} b^2 c^{-3} \int \frac{d\psi}{C^\circ \psi^3} + \frac{3}{8} b^4 c^{-5} \int \frac{d\psi}{C^\circ \psi^5} +$$
ubi $b^2 = 1 - c^2$ & $$\int \frac{d\psi}{C^\circ \psi} = L T^\circ \left(\frac{q+\psi}{2}\right),\ \text{si } q = \frac{\pi}{2} = 90^\circ;$$
$$2\int \frac{d\psi}{C^\circ \psi^3} = L T^\circ \left(\frac{q+\psi}{2}\right) + \frac{S^\circ \psi}{C^\circ \psi^2},$$ atque

Quoniam igitur hæc transformatio maximi est momenti, & via, quam ingressi fuimus, imaginaria nonnullis vel minus lucida vel minus tuta videri possit; hæc aliter explicare juvabit.

Quem in finem nostras functiones iterum geometrice considerabimus, & speciatim ipsas ad amplitudinem illam mediam γ (= ang. $AO\gamma$) referemus.

Quia igitur, ut ex figura & trigonometria patet, $T^0\gamma = \frac{T}{R} = \frac{T}{\sqrt{ab}}$ ideoque $T = \sqrt{ab}\,.\,T^0\gamma$, erit differentiando

$$dT = \frac{\sqrt{ab}\,.\,d\gamma}{C^0\gamma^2} = d\varphi\,.\sqrt{(a^2+T^2)(b^2+T^2)} \text{ seu } dT =$$

$$= d\varphi\,.\sqrt{(a^2+ab\,T^0\gamma^2)\,.\,(b^2+ab\,T^0\gamma^2)},$$

$$dT = \sqrt{ab}\,.\,d\varphi\sqrt{(a+b\,T^0\gamma^2)\,.\,(b+a\,T^0\gamma^2)} =$$

$$dT = \sqrt{ab}\,.\,d\varphi\sqrt{ab(1+T^0\gamma^4)+(a^2+b^2)\,T^0\gamma^2}$$

$$dT = \sqrt{ab}\,.\,d\varphi\sqrt{ab\,.(1+T^0\gamma^2)^2+\overline{a-b}^2\,T^0\gamma^2}, \text{ seu}$$

$$dT = \sqrt{ab}\,.\,d\varphi\,.(1+T^0\gamma^2).\sqrt{ab+\overline{a-b}^2\,.\,\tfrac{1}{4}(S^0 2\gamma)^2}; \text{ ideoque}$$

$$d\varphi = \frac{dT\,.\,C^0\gamma^2}{\sqrt{ab}\,.\sqrt{ab+\frac{1}{4}\overline{a-b}^2\,\overline{S^0 2\gamma}^2}} = \frac{d\gamma}{\sqrt{ab+\frac{1}{4}(a-b)^2\,\overline{S^0 2\gamma}^2}}$$

$$\text{seu } d\varphi = \frac{d\gamma}{\sqrt{ab}\,.\sqrt{1+\frac{\overline{a-b}^2}{4ab}\,.\,\overline{S^0 2\gamma}^2}} = \frac{d\,.\,\widehat{2\gamma}}{\sqrt{4ab+\overline{a-b}^2\,.\,\overline{S^0 2\gamma}^2}}.$$

$8\int\frac{d\psi}{C^0\psi^5} = 3\,L\,T^0\,\frac{q+\psi}{2}+\left(\frac{3}{C^0\psi^2}+\frac{2}{C^0\psi^4}\right)S^0\psi$; fere æquali convergentia hæc series gaudebit, si $b^2 < 0{,}000055$ fuerit. Ex eis vero, quæ in contextu explicantur, facile colligitur, modulos c vel b ad hujusmodi limitem minui posse. In hunc finem sufficiet, alterum adhibere transformationem, quoties $e^2 < \frac{1}{2}$, & alteram, quoties $1 > e^2 > \frac{1}{2}$. Prævie vero (in hoc 2° casu) $C^0\psi > \sqrt{\frac{b}{1+b}}$ faciendum, ponendo $S^0\psi = a\,C^0\varphi : \sqrt{a^2-c^2\,S^0\varphi^2}$.